Section 3.4. Partial Fractions.

Examples.

1. $\displaystyle\int \frac{x^3+2x^2-3x+4}{x+1}\,dx$

2. $\displaystyle\int \frac{x-1}{x^3+3x^2+2x}\,dx$

3. $\displaystyle\int \frac{2}{x(x+1)^2}\,dx$

4. $\displaystyle\int \frac{3x+1}{(x+1)(x^2+1)}\,dx$

Exercises.

1. $\displaystyle\int \frac{x^3+x^2+x}{x-1}\,dx$

   Solution. $\displaystyle\int \frac{x^3+x^2+x}{x-1}\,dx = \int \left(x^2+2x+3+\frac{3}{x-1}\right)dx =$
   $\displaystyle\frac{x^3}{3}+x^2+3x+3\ln|x-1|+C.$

2. $\displaystyle\int \frac{1}{x^2-1}\,dx$

   Solution. $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx = \int \left(\frac{1/2}{x-1}-\frac{1/2}{x+1}\right)dx =$
   $\displaystyle\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C.$

3. $\displaystyle\int \frac{3}{(x^2+1)(x^2+4)}\,dx$

   Solution. $\displaystyle\int \frac{3}{(x^2+1)(x^2+4)}\,dx = \int \left(\frac{1}{x^2+1}-\frac{1}{x^2+4}\right)dx =$
   $\displaystyle\arctan x-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C.$

Other Strategies:
reverse product and quotient rules, completing the square.

Examples.

1. $\displaystyle\int (5x^4 \sin x + x^5 \cos x)\, dx$

2. $\displaystyle\int \frac{x \sec^2 x - \tan x}{x^2}\, dx$

3. $\displaystyle\int \frac{1}{x^2 + 2x + 2}\, dx$

Exercises.

1. $\displaystyle\int e^x (x^4 + 4x^3)\, dx$

   Solution. $\displaystyle\int e^x (x^4 + 4x^3)\, dx = \int (e^x)' \cdot x^4 + e^x \cdot (x^4)'\, dx = \int (e^x \cdot x^4)'\, dx =$ $e^x \cdot x^4 + C = x^4 e^x + C$

2. $\displaystyle\int \frac{\ln x - 1}{\ln^2 x}\, dx$

   Solution. $\displaystyle\int \frac{\ln x - 1}{\ln^2 x}\, dx = \int \frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{(\ln x)^2}\, dx =$
   $\displaystyle\int \frac{(x)' \cdot \ln x - x \cdot (\ln x)'}{(\ln x)^2}\, dx = \int \left(\frac{x}{\ln x}\right)'\, dx = \frac{x}{\ln x} + C$

3. $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\, dx$

   Solution. $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\, dx = \int \frac{1}{\sqrt{1 - (1 - 2x + x^2)}}\, dx =$
   $\displaystyle\int \frac{1}{\sqrt{1 - (x - 1)^2}}\, dx = \arcsin(x - 1) + C$

4. $\displaystyle\int e^x (\cos^2 x + \cos x \sin x - \sin^2 x) dx$

   Solution. $\displaystyle\int e^x (\cos^2 x + \cos x \sin x - \sin^2 x)\, dx = \int (e^x \cos x \sin x)'\, dx =$ $e^x \cos x \sin x + C$

Section 3.6. Numerical Integration.

Exercise. Estimate the value of $\int_0^8 x^2\,dx$ using four subintervals and

(1) left endpoints.

(2) right endpoints.

(3) midpoint rule.

(4) trapezoidal rule.

(5) Simpson's rule.

Find the exact value of the integral. Which of the above methods gave the best approximation?

Solution. Since $\Delta x = 2$, we have

(1) $(0^2 + 2^2 + 4^2 + 6^2) \cdot 2 = 56 \cdot 2 = 112.$

(2) $(2^2 + 4^2 + 6^2 + 8^2) \cdot 2 = 120 \cdot 2 = 240.$

(3) $(1^2 + 3^2 + 5^2 + 7^2) \cdot 2 = 84 \cdot 2 = 168.$

(4) $\frac{1}{2} \cdot (0^2 + 2 \cdot 2^2 + 2 \cdot 4^2 + 2 \cdot 6^2 + 8^2) \cdot 2 = 176.$

(5) $\frac{1}{3} \cdot (0^2 + 4 \cdot 2^2 + 2 \cdot 4^2 + 4 \cdot 6^2 + 8^2) \cdot 2 = \frac{512}{3} = 170.\overline{6}.$

$\int_0^8 x^2\,dx = \left.\frac{x^3}{3}\right|_0^8 = \frac{8^3 - 0}{3} = \frac{512}{3} = 170.\overline{6}$. Simpson's rule gave the exact value. Note that using the left endpoints and the right endpoints resulted in an underestimate and and overestimate because the function is increasing (rather rapidly). The trapezoidal rule gave an overestimate (but not nearly as bad as the right endpoints) because the graph of the function is concave up.

Section 3.7. Improper Integrals.

Examples. Determine whether the following integrals are convergent. If convergent, find the value of the integral.

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
3. $\displaystyle\int_0^1 \frac{1}{x}\,dx$
4. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
6. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2}\,dx$

Exercises. Determine whether the following integrals are convergent. If convergent, find the value of the integral.

1. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$

   Solution. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1} x\Big|_0^t =$ $\displaystyle\lim_{t\to\infty}(\tan^{-1} t - \tan^{-1} 0) = \frac{\pi}{2}.$

2. $\displaystyle\int_0^\infty \cos x\,dx$

   Solution. $\displaystyle\int_0^\infty \cos x\,dx = \lim_{t\to\infty}\int_0^t \cos x\,dx = \lim_{t\to\infty}(\sin t - \sin 0) = \lim_{t\to\infty}\sin t.$ The limit does not exist, so the integral diverges.

3. $\displaystyle\int_{\pi/4}^{\pi/2} \tan x\,dx$

Solution. $\displaystyle\int_{\pi/4}^{\pi/2} \tan x\,dx = \lim_{t\to\pi/2} \int_{\pi/4}^{t} \tan x\,dx = \lim_{t\to\pi/2} \ln|\sec x|\Big|_{\pi/4}^{t} = \lim_{t\to\pi/2} (\ln|\sec t| - \ln|\sec(\pi/4)|) = \infty$, thus the integral diverges.

4. $\displaystyle\int_0^{\infty} xe^x\,dx$

Solution. Integrating by parts, we have
$\displaystyle\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$. Then
$\displaystyle\int_0^{\infty} xe^x\,dx = \lim_{t\to\infty} \int_0^{t} xe^x\,dx = \lim_{t\to\infty} (xe^x - e^x)\Big|_0^t = \lim_{t\to\infty} (te^t - e^t + 1) = \infty$,
thus the integral diverges.

5. $\displaystyle\int_{-\infty}^{0} xe^x\,dx$

Solution. $\displaystyle\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty} \int_t^{0} xe^x\,dx = \lim_{t\to-\infty} (xe^x - e^x)\Big|_t^0 = \lim_{t\to-\infty} (-1 - te^t + e^t) = -1$.