Section 3.1. Integration by parts.

Examples. Evaluate the following integrals.

1. $\displaystyle\int x \sin x \, dx$

2. $\displaystyle\int_0^1 x^2 e^{2x} \, dx$

Exercises. Evaluate the following integrals.

1. $\displaystyle\int \ln x \, dx$ (use $u = \ln x$ and $dv = dx$)

   Solution. We have $du = \frac{1}{x} \, dx$ and $v = x$. Then
   $\displaystyle\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$

2. $\displaystyle\int \frac{\ln x}{x^2} \, dx$

   Solution. Letting $u = \ln x$ and $dv = \frac{1}{x^2} \, dx$, we have $du = \frac{1}{x} \, dx$ and $v = -\frac{1}{x}$. Then $\displaystyle\int \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$

3. $\displaystyle\int_0^1 x^3 e^{-x} \, dx$

   Solution. Let's first evaluate the indefinite integral. Integrate by parts three times: first with $u = x^3$, then $u = x^2$, and finally, with $u = x$. In all cases, $dv = e^{-x} \, dx$, so $v = -e^{-x}$. We have $\displaystyle\int x^3 e^{-x} \, dx =$
   $\displaystyle -x^3 e^{-x} + 3 \int x^2 e^{-x} \, dx = -x^3 e^{-x} - 3x^2 e^{-x} + 6 \int x e^{-x} \, dx =$
   $\displaystyle -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} + 6 \int e^{-x} = -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} + C.$
   Then $\displaystyle\int_0^1 x^3 e^{-x} \, dx = \left(-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x}\right)\Big|_0^1 = -16e^{-1} + 6.$

Section 3.2 Trigonometric Integrals.

Examples.

1. $\displaystyle\int \cos^3 x \sin x \, dx$

2. $\displaystyle\int \cos^3 x \sin^3 x \, dx$

Useful trig identities:
$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x$
$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
$\sin(2x) = 2 \sin x \cos x$
$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$
$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$
$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

Exercises.

1. $\displaystyle\int \cos^2 x \, dx$

   Solution. $\displaystyle\int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$

2. $\displaystyle\int \cos^2 x \sin^2 x \, dx$

   Solution. $\displaystyle\int \cos^2 x \sin^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{8}\int (1 - \cos 4x) \, dx =$
   $\displaystyle\frac{x}{8} - \frac{\sin 4x}{32} + C$
   or $\displaystyle\int \cos^2 x \sin^2 x \, dx = \frac{1}{4}\int (1 + \cos 2x)(1 - \cos 2x) \, dx =$
   $\displaystyle\frac{1}{4}\int (1 - \cos^2 2x) \, dx = \frac{1}{4}\int \left(1 - \frac{1}{2}(1 + \cos 4x)\right) \, dx =$
   $\displaystyle\frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) \, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.$

3. $\int \sin 5x \cos 3x \, dx$

Solution. $\int \sin 5x \cos 3x \, dx = \frac{1}{2} \int (\sin 8x + \sin 2x) \, dx =$
$-\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C.$

4. $\int \tan^5 x \sec^2 x \, dx$

Solution. $\int \tan^5 x \sec^2 x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\tan^6 x}{6} + C,$
where $u = \tan x$.

Section 3.3 Trigonometric Substitution.

Examples.

1. $\int \sqrt{1-x^2}\,dx$

2. $\int \frac{1}{\sqrt{4+x^2}}\,dx$

Recall that

- $1-\sin^2 t = \cos^2 t$,
- $1+\tan^2 t = \sec^2 t$,
- $\sec^2 t - 1 = \tan^2 t$.

So for integrals involving

- $\sqrt{a^2-x^2}$, use $x = a\sin t$,
- $\sqrt{a^2+x^2}$, use $x = a\tan t$,
- $\sqrt{x^2-a^2}$, use $x = a\sec t$.

Exercises.

1. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

   Solution. Although the trig substitution $x = 2\sin t$ can be used (and will work fine), it is not actually needed here since there is a faster way: $\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C.$

2. $\int \frac{1}{\sqrt{x^2-4}}\,dx$

   Solution. Using $x = 2\sec t$, $dx = 2\sec t\tan t\,dt$, we have $\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec t\tan t}{2\tan t}\,dt = \int \sec t\,dt = \ln|\sec t + \tan t| + C =$ $\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C.$

3. $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$

Solution. Using $x = \tan t$, $dx = \sec^2 t\,dt$, we have

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{\sec^2 t}{\sec^4 t}\,dt = \int \cos^2 t\,dt = \frac{1}{2}\int 1 + \cos 2t\,dt =$$

$$\frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{\tan^{-1} x}{2} + \frac{\sin t \cos t}{2} + C = \frac{\tan^{-1} x}{2} + \frac{x}{2(1+x^2)} + C.$$