Quadratic functions and equations

Quadratic formula. The roots of $ax^2 + bx + c = 0$ can be found by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1. For what values of p does $4x^2 + 4px + 4 - 3p = 0$ have two distinct real roots?

Solution. A quadratic equation has two distinct real roots whenever the discriminant $D = b^2 - 4ac$ is positive. Thus we need

$$\begin{aligned}
(4p)^2 - 4 \cdot 4(4 - 3p) &> 0 \\
16p^2 - 16(4 - 3p) &> 0 \\
p^2 - (4 - 3p) &> 0 \\
p^2 + 3p - 4 &> 0 \\
(p + 4)(p - 1) &> 0
\end{aligned}$$

So $p < -4$ or $p > 1$.

Vieta's formulas.

- If $x^2 + bx + c = 0$ has roots (real or complex) r and s, then the polynomial on the left can be factored as $(x - r)(x - s)$. Since $(x - r)(x - s) = x^2 - (r + s)x + rs$, we have

$$r + s = -b,$$
$$rs = c.$$

- If $ax^2 + bx + c = 0$ has roots (real or complex) r and s, then we can write the equation in the monic form by dividing by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

therefore

$$r + s = -\frac{b}{a},$$
$$rs = \frac{c}{a}.$$

Example 2. (2002 AMC 12A #12) Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. Find the number of possible values of k.

Solution. If the roots are r and s, we have $r + s = 63$ and $rs = k$. The first equation implies that $\{r, s\} = \{2, 61\}$, so $k = rs = 2 \cdot 61 = 122$, thus there is only one such value of k.

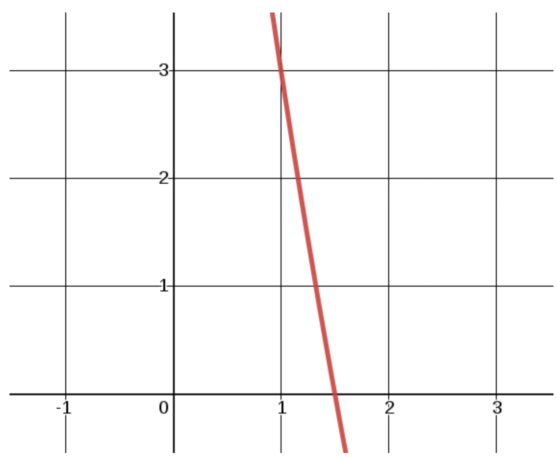
More observations about quadratic functions and their graphs.
When the equation $ax^2 + bx + c = 0$ has two real roots, we see from the quadratic formula that the value $-\dfrac{b}{2a}$ is their average, therefore it is the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$.
It can be shown that the x-coordinate of the vertex of the parabola $y = ax^2+bx+c$ is always $-\frac{b}{2a}$, even if the roots (also called zeros of the function) are not real (if the discriminant $D = b^2 - 4ac$ is negative).

Also, for $f(x) = ax^2 + bx + c$,

$$f(0) = c$$
$$f(1) = a + b + c$$
$$f(-1) = a - b + c$$

(Note also that these formulas can be generalized for polynomials of any degree.)

Example 3. A small fragment of the curve $y = x^2 + bx + c$ is shown below (if it seems to pass through a certain grid point, assume that it does). Find the value of $b + c$.



Solution. We see from the graph that $y(1) = 3$, so $1 + b + c = 3$. Therefore $b + c = 2$.

Fresno Math Circle **Grades 8-9**

Quadratic functions and equations

1. Find all values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x.

2. For how many real numbers a does the quadratic equation $x^2 + ax + 123 = 0$ have two integer roots?

3. Let a, b, c, and d be four distinct one-digit numbers. What is the maximum possible value of the sum of the roots of the equation $(x-a)(x-b)+(x-c)(x-d)=0$?

4. Suppose $p(x)$ is a polynomial of degree 2, has roots 3 and -5, and $p(0)=-60$. What is the coefficient of x in $p(x)$?

5. (2007 AMC 12A #21) The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

   (a) the coefficient of x^2

   (b) the coefficient of x

   (c) the y-intercept of the graph $y = f(x)$

   (d) one of the x-intercepts of the graph of $y = f(x)$

   (e) the mean of the x-intercepts of the graph of $y = f(x)$

6. (2005 AMC 10B #16, AMC 12B #12) The quadratic equation $x^2 + mx + n = 0$ has roots twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of n/p?

7. (2006 AMC 10B #14) Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

8. (2003 AMC 10A #18) What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

9. Let us take another look at the quadratic function $f(x) = 4x^2 + 4px + 4 - 3p$. Suppose that for every real value of p, we graph the parabola $y = 4x^2 + 4px + 4 - 3p$, and mark its vertex. What curve do these vertices form?

10. If α and β are solutions of $x^2 + px + q = 0$, find $\alpha^2 + \beta^2$ in terms of p and q.

11. A quadratic polynomial f satisfies $f(x) \geq 2$ for all x, $f(2) = 2$, and $f(3) = 4$. What is $f(5)$?

12. (2015 AMC 10A #23) The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

13. (2011 AMC 10B #19) What is the product of all the roots of the equation

$$\sqrt{5|x|+8} = \sqrt{x^2-16}?$$

14. (2014 AMC 10B #20) For how many integers x is the number $x^4 - 51x^2 + 50$ negative?

15. (2013 AMC 10B #19) The real numbers c, b, a form an arithmetic sequence with

$$a \geq b \geq c \geq 0.$$

The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

16. (1977 Canadian MO) If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.

17. Suppose that the absolute values of the real roots of

$$x^2 + Ax + B = 0$$

and

$$x^2 + Cx + D = 0$$

are less than 1. Prove that the absolute values of the real roots (if there are any) of

$$x^2 + \frac{A+C}{2}x + \frac{B+D}{2} = 0$$

are also less than 1.

18. (2011 AIME #6) Suppose that a parabola has vertex $\left(\frac{1}{4}, -\frac{9}{8}\right)$ and equation

$$y = ax^2 + bx + c,$$

where $a > 0$ and $a + b + c$ is an integer. Find the minimum possible value of a.

19. (2020 AMC 12A #25) There exists a real number a such that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is the value of a?

Instructor Notes

Quadratic functions and equations: answers and solutions

1. The quadratic equation has only one solution when the discriminant

$$D = (a+8)^2 - 4 \cdot 4 \cdot 9 = 0,$$

that is, $a + 8 = \pm 12$. We have two solutions: $a = 4$ and $a = -20$.

2. The product of the roots is 123. Since the prime factorization of 123 is $3 \cdot 41$, the two roots can be 3 and 41, or -3 and -41, or 1 and 123, or -1 and -123. The coefficient a is the additive inverse of the sum of the roots, so there are 4 possible values (-44, 44, -124, and 124, respectively).

3. Expanding the products and combining like terms we have

$$2x^2 - (a+b+c+d)x + (ab+cd) = 0.$$

The sum of the roots is $\frac{a+b+c+d}{2}$, so we need to find the maximum possible value of this expression given that a, b, c, and d be four distinct one-digit numbers. The largest possible four distinct one-digit numbers are 9, 8, 7, and 6, so the largest possible value is $\frac{9+8+7+6}{2} = 15$.

4. Solution 1. Let $p(x) = ax^2 + bx + c$. From the given information we conclude that

$$-\frac{b}{a} = 3 + (-5) = -2,$$
$$\frac{c}{a} = 3(-5) = -15,$$
$$c = p(0) = -60.$$

It follows that $a = \frac{c}{-15} = 4$, and the coefficient of x is $b = 2a = 8$.

Solution 2. Since 3 and -5 are roots of the polynomial, it can be factored as

$$p(x) = a(x-3)(x+5).$$

Then $p(0) = -15a$, so $a = 4$. Then $p(x) = 4(x-3)(x+5) = 4(x^2+2x-15) = 4x^2 + 8x - 60$, so the coefficient of x is 8.

5. The sum of the zeros (roots) is $-\frac{b}{a}$, their product is $\frac{c}{a}$, and the sum of the coefficients is $a+b+c$. So we have

$$-\frac{b}{a} = \frac{c}{a} = a+b+c.$$

It follows that $c = -b$, and thus

$$a+b+c = a,$$

thus their common value is equal to the coefficient of x^2.

6. If the roots of $x^2 + mx + n = 0$ are twice those of $x^2 + px + m = 0$, then the sum of the roots of the first equation, $-m$, is also twice the sum of the roots of the second equation, $-p$, so we have $-m = 2(-p)$, that is, $m = 2p$. Also, the product of the roots of the first equation, n, is four times larger than the product of the roots of the second equation, m, so $n = 4m$. It follows that $n = 4m = 8p$, so $n/p = 8$.

7. From the first equation we know that $ab = 2$. From the second equation,

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = \frac{ab+1}{b} \cdot \frac{ab+1}{a} = \frac{(ab+1)^2}{ab} = \frac{9}{2}.$$

Equivalently,

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 2 + \frac{1}{ab} = \frac{9}{2}.$$

8. Solution 1. First let us multiply the equation by x to obtain a quadratic equation:

$$\frac{2003}{2004}x^2 + x + 1 = 0.$$

Let r and s be its roots. Then $r + s = -\frac{2004}{2003}$ and $rs = \frac{2004}{2003}$. The sum of their reciprocals is

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = -1.$$

Solution 2. The reciprocals of the roots of an equation are the roots of the equation obtained by replacing each x with $\frac{1}{x}$ in the original equation. So, the sum we want to find is just the sum of the roots of the equation $\frac{2003}{2004} \cdot \frac{1}{x} + 1 + x = 0$. Multiplying this equation by x gives a quadratic equation:

$$x^2 + x + \frac{2003}{2004} = 0,$$

and the sum of its roots is -1.

9. The vertex of the parabola has x-coordinate $-\frac{b}{2a} = -\frac{4p}{8} = -\frac{p}{2}$ and y-coordinate $y\left(-\frac{p}{2}\right) = 4\left(-\frac{p}{2}\right)^2 + 4p\left(-\frac{p}{2}\right) + 4 - 3p = 4\left(-\frac{p}{2}\right)^2 - 8\left(-\frac{p}{2}\right)^2 - 3p + 4 = -4\left(-\frac{p}{2}\right)^2 + 6\left(-\frac{p}{2}\right) + 4$, so the vertex lies on the parabola $y = -4x^2 + 6x + 4$.

Remark. We recommend graphing both quadratics, $y = 4x^2 + 4px + 4 - 3x$ with a slider for p, e.g. from -10 to 10, and $y = -4x^2 + 6x + 4$. As we change the value of p, we see that the first parabola moves, but its vertex is always on the second parabola.

10. If α and β are solutions of $x^2 + px + q = 0$, we have $\alpha + \beta = -p$ and $\alpha\beta = q$. Then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-p)^2 - 2q = p^2 - 2q$.

11. Solution 1. If $f(x) \geq 2$ for all x and $f(2) = 2$, then the vertex of the parabola is at $(2, 2)$, so $f(x) = a(x-2)^2 + 2$. Then $4 = f(3) = a + 2$, so $a = 2$, and thus $f(x) = 2(x-2)^2 + 2$. Then $f(5) = 2 \cdot 3^2 + 2 = 20$.

Solution 2. Let $f(x) = ax^2 + bx + c$. The first step is the same as in Solution 1: since $f(x) \geq 2$ for all x and $f(2) = 2$, then the vertex of the parabola is at $(2, 2)$. It follows that

$$
\begin{aligned}
-\frac{b}{2a} &= 2 \\
4a + 2b + c &= f(2) = 2 \\
9a + 3b + c &= f(3) = 4.
\end{aligned}
$$

Subtracting the second equation from the third we get $5a+b = 2$. From the first equation $b = -4a$, therefore $a = 2$. Then $b = -8$ and $c = 10$. Thus $f(x) = 2x^2 - 8x + 10$ and then $f(5) = 20$.

12. Let r and s be the zeros, then $r + s = a$ and $rs = 2a$. Then

$$
\begin{aligned}
2r + 2s &= rs \\
rs - 2r - 2s &= 0 \\
rs - 2r - 2s + 4 &= 4 \\
(r - 2)(s - 2) &= 4
\end{aligned}
$$

The right-hand side, 4, has four possible factorizations: $2 \cdot 2$, $(-2) \cdot (-2)$, $1 \cdot 4$, and $(-1) \cdot (-4)$, so this gives four possible (unordered) pairs of roots r and s: 4 and 4, 0 and 0, 3 and 6, 1 and -2. Thus the possible values of a are 8, 0, 9, and -1. Their sum is $8 + 0 + 9 + (-1) = 16$.

13. Let us square both sides of the equation:

$$
\begin{aligned}
5|x| + 8 &= x^2 - 16 \\
x^2 - 5|x| - 24 &= 0
\end{aligned}
$$

If $x \geq 0$, then we have

$$
\begin{aligned}
x^2 - 5x - 24 &= 0 \\
(x - 8)(x + 3) &= 0.
\end{aligned}
$$

Only the root $x = 8$ satisfies the condition $x \geq 0$ (and thus the original equation). If If $x < 0$, then we have

$$
\begin{aligned}
x^2 + 5x - 24 &= 0 \\
(x + 8)(x - 3) &= 0.
\end{aligned}
$$

Only the root $x = -8$ satisfies the condition $x < 0$ (and thus the original equation).

Thus the product of all the roots of the equation $\sqrt{5|x| + 8} = \sqrt{x^2 - 16}$ is -64.

14. Observe that $x^4 - 51x^2 + 50 = (x^2 - 50)(x^2 - 1)$. This value is negative when $1 < x^2 < 50$, that is, $2 \leq |x| \leq 7$. There are six positive and six negative values, so 12 values total.

15. Since c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$, $2b = a + c$, so $c = 2b - a$. If the quadratic $ax^2 + bx + c$ has exactly one root, then $b^2 - 4ac = 0$. Substituting $2b - a$ for c, we have $b^2 - 4a(2b - a) = 0$, or, equivalently, $b^2 - 8ab + 4a^2 = 0$. Then $\left(\frac{b}{a}\right)^2 - 8 \cdot \frac{b}{a} + 4 = 0$, so $\frac{b}{a} = 4 \pm 2\sqrt{3}$. Since $a > b > 0$, we have $\frac{b}{a} < 1$, so $\frac{b}{a} = 4 - 2\sqrt{3}$. Then the only root is $-\frac{b}{2a} = -2 + \sqrt{3}$.

16. Let us rewrite the given equation as

$$\begin{aligned} 4a^2 + 4a &= b^2 + b \\ 4a^2 + 4a + 1 &= b^2 + b + 1 \\ (2a + 1)^2 &= b^2 + b + 1 \end{aligned}$$

Since $b^2 < b^2 + b + 1 < b^2 + 2b + 1 = (b + 1)^2$, we see that $b^2 + b + 1$ cannot be a perfect square.

17. If $|x| \geq 0$, then

$$x^2 + Ax + B > 0$$

and

$$x^2 + Cx + D > 0.$$

It follows that

$$x^2 + \frac{A + C}{2}x + \frac{B + D}{2}x = \frac{(x^2 + Ax + B) + (x^2 + Cx + D)}{2} > 0.$$

18. Let us use the formula for the x-coordinate of a vertex of a parabola: $-\frac{b}{2a} = \frac{1}{4}$. Solving this equation gives $-\frac{a}{2} = b$. Then $-\frac{9}{8} = y\left(\frac{1}{4}\right) = \frac{a}{16} + \frac{b}{4} + c = -\frac{a}{16} + c$, so $c = \frac{a-18}{16}$. This means that $\frac{9a-18}{16} = a + b + c \in Z$. Since $\frac{9a-18}{16} > -2$, the minimum value of $a > 0$ is when $\frac{9a-18}{16} = -1$, thus $a = \frac{2}{9}$.