MATH 149S

Study Guide and Sample Problems for Test 1

Note: the actual test will consist of five questions, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Divisibility and congruences
   - (a) State and prove divisibility tests for 2, 3, 4, 5, 8, 9, 10, and 11.
   - (b) Prove that if $a|b$ and $a|c$, then $a|(b+c)$.
   - (c) Prove that $a \equiv b \pmod{10}$ if and only if a and b have the same units digit.
   - (d) Can a perfect square end with the digits 154?
2. Bases
   - (a) When a positive integer A is written in base 5, it has exactly the same digits (which are in the same order) as a positive integer B written in base 6. Which is larger, A or B?
   - (b) An integer x has 3 digits when written in base 10. How many digits can it have when written in base 3?
   - (c) Convert 2000_4 to base 7.
   - (d) How many 4-digit numbers in base 6 are there?
   - (e) State and prove divisibility tests for $b-1$, b, and $b+1$ for numbers written in base b.
3. Combinatorics
   - (a) Explain why there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose k objects out of n.
   - (b) Prove that $\binom{n}{k} = \binom{n}{n-k}$.
   - (c) Prove that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.
   - (d) How many positive factors does 100000 have?
   - (e) How many even positive factors does 100000 have?
   - (f) How many of the positive factors of 100000 are perfect squares? Perfect cubes?

(g) Explain why there are $\binom{n+k-1}{n}$ ways to distribute n identical balls into k distinguishable boxes.

(h) How many ways are there to distribute n identical balls into k distinguishable boxes if each box should contain at least one ball? At least two balls?

(i) How many ways are there to distribute n distinguishable balls into k distinguishable boxes?

(j) How many positive integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ have?

4. Binomial theorem

(a) Expand: $(x + y)^n$.

(b) Find the first three terms in the expansions of $(2x + 1)^5$, $(x - 2y)^{10}$.

(c) Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

5. Probability, discrete and continuous variables.

(a) There are 5 white, 6 red, and 7 blue balls in a bag. Two balls are drawn randomly. What is the probability that they are both blue?

(b) How many ways are there to choose 3 cards from a deck of 52 cards? How many ways are there to choose 3 cards from the 12 "face" cards (J, Q, K)?

(c) If three cards are chosen randomly for a deck of 52 cards, what is the probability that all three are face cards?

(d) Two numbers, x and y, are randomly chosen in the interval $[0, 1]$. What is the probability that $2x + y > 1$?

(e) A stick is broken at two random places. What is the probability that the longest piece is at least $\frac{3}{4}$ of the stick's length?