MATH 111

Practice Final – Solutions

1. Prove or disprove the following statement:
   Let A, B, and C be sets. Then $(A \cup B) - C = (A - C) \cup (B - C)$.

   The statement is true.

   First we will prove that $(A \cup B) - C \subset (A - C) \cup (B - C)$. *Let* $x \in (A \cup B) - C$. *Then* $x \in A \cup B$ *and* $x \notin C$. *Then* $x \in A$ *or* $x \in B$, *and* $x \notin C$.
   Case I: $x \in A$. *Since* $x \notin C$, $x \in A - C$. *Therefore* $x \in (A - C) \cup (B - C)$.
   Case II: $x \in B$. *Since* $x \notin C$, $x \in B - C$. *Therefore* $x \in (A - C) \cup (B - C)$.

   Next we will prove that $(A - C) \cup (B - C) \subset (A \cup B) - C$. *Let* $x \in (A - C) \cup (B - C)$. *Then* $x \in A - C$ *or* $x \in B - C$.
   Case I: $x \in A - C$. *Then* $x \in A$ *and* $x \notin C$. *Then* $x \in A \cup B$ *and* $x \notin C$, *so* $x \in (A \cup B) - C$.
   Case II: $x \in B - C$. *Then* $x \in B$ *and* $x \notin C$. *Then* $x \in A \cup B$ *and* $x \notin C$, *so* $x \in (A \cup B) - C$.

2. Determine whether the compound propositions $(P \vee Q) \Rightarrow (P \wedge Q)$ and $P \Leftrightarrow Q$ are logically equivalent.

   We construct the truth table for these two propositions:

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \Rightarrow (P \wedge Q)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

   Since for all truth values of P *and* Q *the propositions* $(P \vee Q) \Rightarrow (P \wedge Q)$ *and* $P \Leftrightarrow Q$ *have the same truth value, they are logically equivalent.*

3. Let $n \in \mathbb{Z}$. Prove that if $3n^2 + 4n + 2$ is even, then n is even.

   We will prove this statement by contrapositive. If n *is odd, then* $n = 2k + 1$ *for some* $k \in \mathbb{Z}$. *Then* $3n^2 + 4n + 2 = 3(2k+1)^2 + 4(2k+1) + 2 = 3(4k^2 + 4k + 1) + 8k + 4 + 2 = 12k^2 + 20k + 9 = 2(6k^2 + 10k + 4) + 1$. *Since* $6k^2 + 10k + 4 \in \mathbb{Z}$, $3n^2 + 4n + 2$ *is odd.*

4. Prove or disprove the following statement:
   For any $a \in \mathbb{Z}$, the number $a^3 + a + 100$ is positive.

The statement is false. For $a = -5$, $a^3+a+100 = -125-5+100 = -30$ *is not positive.*

5. Consider the relation R defined on $\mathbb{Z}$ by aRb iff $ab \leq 0$. Determine whether R is
   (a) reflexive,

   This relation is not reflexive: e.g. $1 \cdot 1 \not\leq 0$, *so* $(1, 1) \notin R$.

   (b) symmetric,

   It is symmetric: if $(a, b) \in R$, *then* $ab \leq 0$, *then* $ba = ab \leq 0$, *so* $(b, a) \in R$.

   (c) transitive,

   It is not transitive: consider, e.g., $1 \cdot 0 \leq 0$, $0 \cdot 1 \leq 0$, *but* $1 \cdot 1 \not\leq 0$, *so* $(1, 0) \in R$, $(0, 1) \in R$, *but* $(1, 1) \in R$.

   (d) an equivalence relation.

   It is not an equivalence relation since it is not reflexive.

6. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$

   Determine whether f is
   (a) injective,

   The function f *is not injective: e.g.* $f(1) = 2 = f(2)$, *but* $1 \neq 2$.

   (b) surjective,

   We will prove that for any $x \in \mathbb{Z}$, $f(x)$ *is even.*
   Case I: x *is even. Then* $f(x) = x$ *is even.*
   Case II: x *is odd. Then* $f(x) = 2x$ *is even (since* $x \in \mathbb{Z}$*).*
   Since 1 *is not even,* $f(x) \neq 1$ *for any* $x \in \mathbb{Z}$, *so* 1 *is not in the image.*
   Thus the function f *is not surjective.*

   (c) bijective.

   The function is not bijective since it is not injective.

7. Prove that the number 111 cannot be written as the sum of four integers, two of which are even and two of which are odd.

   We will prove this statement by contradiction. Assume that the number 111 *can be written as the sum of four integers, two of which are even and two of which are odd. Let* $111 = a + b + c + d$ *where* a *and* b *are even and* c *and* d *are odd. Then* $a = 2k$, $b = 2l$, $c = 2m + 1$, *and* $d = 2n + 1$ *for some* $k, l, m, n \in \mathbb{Z}$. *Then* $111 = 2k + 2l + 2m + 1 + 2n + 1 = 2(k + l + m + n + 1)$. *Since* $k + l + m + n + 1 \in \mathbb{Z}$, 111 *is even. Contradiction.*

8. The following problem could be a problem for extra credit on the final exam: Give an example of a bijective function $f : \mathbb{Z} \to \mathbb{N}$ and find its inverse.

   Let $f : \mathbb{Z} \to \mathbb{N}$ *be defined by* $f(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x + 1 & \text{if } x \leq 0 \end{cases}$ *We will prove that* f *is bijective.*
   Note that the image of every positive integer is even, and the image of every nonpositive integer is odd.
   Let $f(a) = f(b)$.
   Case I: $f(a)$ *is even. Then* $a > 0$ *and* $b > 0$. *Therefore* $f(a) = 2a$ *and* $f(b) = 2b$. *It follows that* $2a = 2b$, *so* $a = b$.
   Case II: $f(a)$ *is odd. Then* $a \leq 0$ *and* $b \leq 0$. *Therefore* $f(a) = -2a + 1$ *and* $f(b) = -2b + 1$. *It follows that* $-2a + 1 = -2b + 1$, *so* $a = b$.
   Thus f *is injective.*
   Now let $b \in \mathbb{N}$. *We will show that there exists* $a \in \mathbb{Z}$ *such that* $f(a) = b$.
   Case I: b *is even. Let* $a = \dfrac{b}{2}$. *Since* b *is even,* $a \in \mathbb{Z}$. *Since* $b > 0$, $a > 0$, *so* $f(a) = 2a = b$.
   Case II: b *is odd. Then* $b = 2k + 1$ *for some* $k \in \mathbb{Z}$. *Let* $a = -k$. *Since* $b \geq 1$, $2k \geq 0$, *so* $a \leq 0$. *Then* $f(a) = -2a + 1 = 2k + 1 = b$.
   Thus f *is surjective.*

   The inverse of f *is* $f^{-1} = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{1-x}{2} & \text{if } x \text{ is odd} \end{cases}$
   To verify that this function is the inverse of f, *we check* $f^{-1} \circ f(x) = x$. *We will consider two cases.*
   Case I: $x > 0$. *Then* $f(x) = 2x$ *is even, so* $f^{-1}(f(x)) = \dfrac{2x}{2} = x$.
   Case II: $x \leq 0$. *Then* $f(x) = -2x + 1$ *is odd, so* $f^{-1}(f(x)) = \dfrac{1 - (-2x + 1)}{2} = \dfrac{1 + 2x - 1}{2} = x$.