# *Symmetry of the power sum polynomials*

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# Outline

- Power sum polynomials
- History
- Recursive definition
- Identity involving Bernoulli numbers
- Symmetry
- Open questions

# Power sum polynomials

Recall these familiar formulas from Calculus:

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

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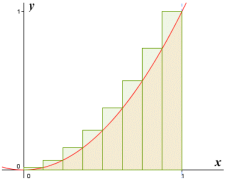

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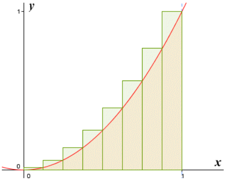
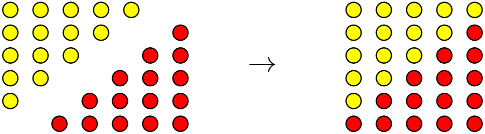



$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

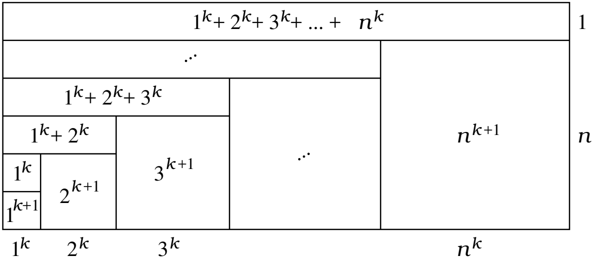
# History

- Pythagoreans (c. 570-500 BCE), Greece



- Abu Ali al-Hasan (965-1039), Egypt

$$(n+1)\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n}\sum_{i=1}^{p} i^k$$



# History (cont.)

- Pascal (1623-1662), France

$$(n+1)^{p+1} - \left(1 + n + \binom{p+1}{2} \sum_{k=1}^{n} k^{p-1} + \binom{p+1}{3} \sum_{k=1}^{n} k^{p-2} + \ldots + (p+1) \sum_{k=1}^{n} k \right) = (p+1)(1^p + 2^p + 3^p + \ldots + n^p)$$

- 1900s
  If $n$ is prime,

$$1^p + 2^p + \ldots + n^p \equiv \begin{cases} -1 \pmod{n} & \text{if } n-1 \mid p \\ 0 \pmod{n} & \text{if } n-1 \nmid p \end{cases}$$

# Recursive definition

## Definition 1

For $n \in \mathbb{R}$, let $S_1(n) = \frac{n(n+1)}{2}$.

For $p \geq 2$ and $n \in \mathbb{R}$, we define

$$S_p(n) = \frac{1}{p+1}\left[(n+1)((n+1)^p - 1) - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i(n)\right].$$

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## Theorem 2

*For* $p, n \in \mathbb{N}$, $S_p(n) = \sum_{k=1}^{n} k^p$.

# Bernoulli numbers

Bernoulli numbers, $B_m$, are defined as follows:

**Definition 3**

Let $B_0 = 1$, and for each $m \geq 1$,

$$\sum_{i=0}^{m} \binom{m+1}{i} B_i = 0.$$

The first few Bernoulli numbers are:

$$1,\ -\frac{1}{2},\ \frac{1}{6},\ 0,\ -\frac{1}{30},\ 0,\ \frac{1}{42},\ \ldots$$

Note that for $m \geq 3$ odd, $B_m = 0$.

# Identity involving Bernoulli numbers

**Theorem 4**

*For* $m, k \in \mathbb{Z},\ m \geq 1,\ 0 \leq k \leq m,$

$$(-1)^{m-k}\binom{m}{k}B_{m-k} = \sum_{i=k}^{m}\binom{m}{i}\binom{i}{k}B_{m-i}$$

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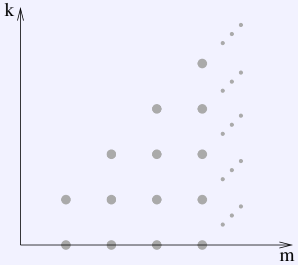
## Sketch of Proof

Induction on both $m$ and $k$

- Consider the case $(m, k) = (m, m)$
- Consider the case $(m, k) = (m, 0)$
- Assume the statement holds for $(m, k)$ and show it holds for $(m + 1, k + 1)$



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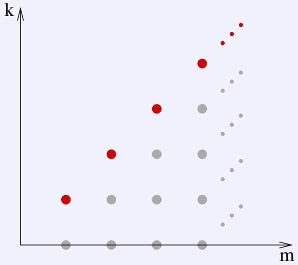

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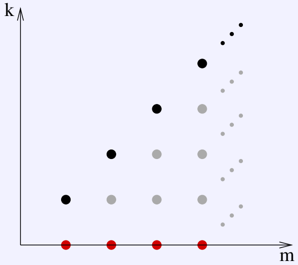

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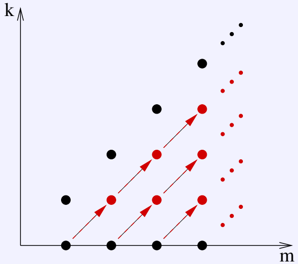
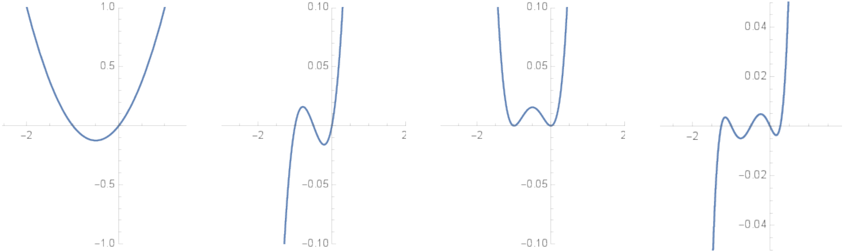


# Symmetry of power sum polynomials

## Theorem 5

*For each $p \in \mathbb{N}$, $S_p(n)$ has symmetry about $-\frac{1}{2}$. Namely, it is symmetric about the vertical line at $-\frac{1}{2}$ if $p$ is odd, and symmetric about the point $(-\frac{1}{2}, 0)$ if $p$ is even.*




$p = 1$ $\quad$ $p = 2$ $\quad$ $p = 3$ $\quad$ $p = 4$

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## Sketch of Proof

**Faulhaber's (Bernoulli's) Formula**

$$S_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} (-1)^i \binom{p+1}{i} B_i n^{p+1-i}$$

Expanding $S_p(-(n+1))$ using the binomial theorem, combining like terms, and using the previous identity for Bernoulli numbers yields

$$S_p(-(n+1)) = \begin{cases} S_p(n) & \text{if } p \text{ is odd,} \\ -S_p(n) & \text{if } p \text{ is even.} \end{cases}$$

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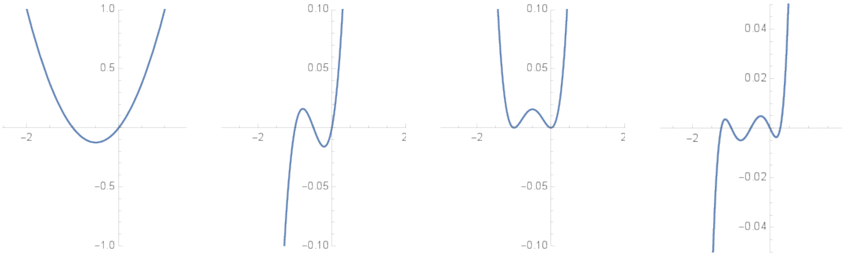


## Corollary 6

*For each $p \in \mathbb{N}$, the roots of $S_p(n)$ are symmetric about $-\frac{1}{2}$. When $p$ is even, $S_p(n)$ has $-\frac{1}{2}$ as a root.*

# Open questions

- How many (distinct) real roots does $S_p(n)$ have?
- Where are the real roots located?
- Where are the complex roots located?

Thank you!